Lower bound for the maximal number of facets of a 0/1 polytope

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Abstract

Let $f_{n-1}(P)$ denote the number of facets of a polytope P in $\mathbb{R}^n$. We show that there exist 0/1 polytopes P with

$$f_{n-1}(P) \geq \left(\frac{cn}{\log^2 n}\right)^{n/2}$$

where $c > 0$ is an absolute constant. This improves earlier work of Bárány and Pór on a question of Fukuda and Ziegler.

1 Introduction

The aim of this article is to give a lower bound for the maximal possible number of facets of a 0/1 polytope in $\mathbb{R}^n$. By definition, a 0/1 polytope is the convex hull of a subset of the vertices of $[0,1]^n$.

In general, if P is a polytope in $\mathbb{R}^n$, we write $f_{n-1}(P)$ for the number of its facets. Let

$$g(n) := \max\left\{f_{n-1}(P_n) : P_n \text{ a } 0/1 \text{ polytope in } \mathbb{R}^n\right\}. \tag{1.1}$$

Fukuda and Ziegler (see [6], [9], [13]) asked what the behaviour of $g(n)$ is as $n \to \infty$. The best known upper bound to date is

$$g(n) \leq 30(n-2)! \tag{1.2}$$

(for n large enough), which is established by Fleiner, Kaibel and Rote in [7]. We will study lower bounds. A major breakthrough in this direction was made by Bárány and Pór in [1]; they proved that

$$g(n) \geq \left(\frac{cn}{\log n}\right)^{n/4}, \tag{1.3}$$

where $c > 0$ is an absolute constant. We will show that the exponent $n/4$ can in fact be improved to $n/2$:

Theorem 1.1 *There exists a constant $c > 0$ such that*

$$g(n) \geq \left(\frac{cn}{\log^2 n}\right)^{n/2}. \tag{1.4}$$

It is interesting to compare this estimate with the known bounds for the expected number of facets of the convex hull $P_{N,n}$ of N independent random points which are uniformly distributed on the sphere S^{n-1}. In [2] it is shown that there exist two constants $c_1, c_2 > 0$, such that

$$\left(c_1 \log \frac{N}{n}\right)^{n/2} \leq \mathbb{E}[f_{n-1}(P_{N,n})] \leq \left(c_2 \log \frac{N}{n}\right)^{n/2} \tag{1.5}$$

for all n and N satisfying $2n \leq N \leq 2^n$. In the case of 0/1 polytopes, N can be as large as 2^n, therefore one might conjecture that $g(n)$ is of the order of $n^{n/2}$. Theorem 1.1 gives a lower bound which is "practically of this order": for every $\varepsilon > 0$ one has

$$g(n) > n^{(0.5-\varepsilon)n} \tag{1.6}$$

if n is large enough.

The existence of 0/1 polytopes with many facets will be established by a refinement of the probabilistic method developed in [1]. It will be more convenient to work with ± 1 polytopes (i.e., polytopes whose vertices are sequences of signs). Let $X_1, \ldots, X_n$ be independent and identically distributed ± 1 random variables, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with distribution

$$\mathbb{P}(X = 1) = \mathbb{P}(X = -1) = \tfrac{1}{2}.$$

Set $\vec{X} = (X_1, \ldots, X_n)$ and, for a fixed N satisfying $n < N \leq 2^n$, consider N independent copies $\vec{X}_1, \ldots, \vec{X}_N$ of $\vec{X}$. This procedure defines the random 0/1 polytope

$$K_N = \mathrm{conv}\{\vec{X}_1, \ldots, \vec{X}_N\}. \tag{1.7}$$

Note that K_N has at most N vertices (it may happen that some repetitions occur).

Under some restrictions on the range of values of N, we will obtain a lower bound for the expected number of facets $\mathbb{E}[f_{n-1}(K_N)]$, for each fixed N. In particular, we have:

Theorem 1.2 *There exist two positive constants a and b such that: for all sufficiently large n, and all N satisfying $n^a \leq N \leq \exp(bn/\log n)$, there exists a 0/1 polytope K_N in $\mathbb{R}^n$ with*

$$f_{n-1}(K_N) \geq \left(\frac{\log N}{a \log n}\right)^{n/2}. \tag{1.8}$$

It is clear that Theorem 1.1 follows: one only has to choose $N = \lfloor \exp(bn/\log n) \rfloor$.

2 Preliminaries

We first fix some standard notation. We work in $\mathbb{R}^n$ which is equipped with a Euclidean structure $\langle\cdot,\cdot\rangle$. We denote by $\|\cdot\|_2$ the corresponding Euclidean norm, by $\|\cdot\|_\infty$ the max-norm, and write B_2^n for the Euclidean unit ball and S^{n-1} for the unit sphere. Volume, surface area and the cardinality of a finite set are denoted by $|\cdot|$ (this will cause no confusion). All logarithms are natural. Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leq b \leq c_2 a$. The letters c, c', C, c_1, c_2 etc. denote absolute positive constants which may change from line to line (however, in most places we will try to specify the absolute constants involved).

As mentioned in the introduction, the proof of Theorem 1.2 will be modeled on the approach of Bárány and Pór. This in turn has its origin in the work of Dyer, Füredi and McDiarmid [3], who proved the following: Let $\kappa = 2/\sqrt{e}$ and consider the random polytope K_N defined in (1.7). For every $\varepsilon \in (0,1)$,

$$\lim_{n\to\infty} \sup\left\{2^{-n}\mathbb{E}|K_N| \colon N \leq (\kappa-\varepsilon)^n\right\} = 0 \tag{2.1}$$

and

$$\lim_{n\to\infty} \inf\left\{2^{-n}\mathbb{E}|K_N| \colon N \geq (\kappa+\varepsilon)^n\right\} = 1. \tag{2.2}$$

In order to determine the threshold $N(n) = (2/\sqrt{e})^n$, they introduced two families of convex subsets of the cube $C = [-1,1]^n$. For every $\vec{x} \in C$, set

$$q(\vec{x}) := \inf\left\{\mathrm{Prob}\big(\vec{X} \in H\big) \colon \vec{x} \in H,\ H \text{ a closed halfspace}\right\}. \tag{2.3}$$

If $\beta > 0$ then the β-center of C is defined by

$$Q^\beta = \{\vec{x} \in C \colon q(\vec{x}) \geq \exp(-\beta n)\}; \tag{2.4}$$

it is easily checked that Q^β is a convex polytope.

Next, consider the function $f : (-1,1) \to \mathbb{R}$ with

$$f(x) = \tfrac{1}{2}(1+x)\log(1+x) + \tfrac{1}{2}(1-x)\log(1-x), \tag{2.5}$$

extend it to a continuous function on $[-1,1]$ by setting $f(\pm 1) = \log 2$, and for every $\vec{x} = (x_1, \ldots, x_n) \in C$ set

$$F(\vec{x}) = \frac{1}{n}\sum_{i=1}^{n} f(x_i). \tag{2.6}$$

The next lemma was proved in [3, Section 3].

Lemma 2.1 *For every $\vec{x} \in (-1,1)^n$ we have $q(\vec{x}) \leq \exp(-nF(\vec{x}))$.* $\square$

The second family of subsets of C introduced in [3] is as follows: for every $\beta > 0$, set

$$F^{\beta} = \{\vec{x} \in C : F(\vec{x}) \leq \beta\}. \tag{2.7}$$

Since f is a strictly convex function on $(-1, 1)$, it is clear that F^{β} is convex. Lemma 2.1 and the definition of Q^{β} show that if $\vec{x} \in Q^{\beta} \cap (-1, 1)^n$ then $F(\vec{x}) \leq \beta$. In other words, we have the following.

Lemma 2.2 *$Q^{\beta} \cap (-1, 1)^n \subseteq F^{\beta}$ for every $\beta > 0$.* □

Observe that as $\beta \to \log 2$, both Q^{β} and F^{β} approach C. The main technical step for the proof of Theorem 1.2 will be to show that the two families are very close, in the following sense: for a wide range of β's, one has that

$$F^{\beta-\varepsilon} \cap \gamma C \subseteq Q^{\beta} \cap (-1, 1)^n \subseteq F^{\beta}, \tag{2.8}$$

where $\gamma > 0$ is a (small) absolute constant and $\varepsilon \leq 3 \log n/n$. The estimate on ε substantially improves (and at the same time clarifies) [1, Lemma 4.3] and should be viewed as the main technical step in our work. The proof is presented in the next Section. It is based on two ingredients: (1) Theorem 3.3, which via large deviations exhibits, for each $\vec{x} \in \partial(F^{\beta})$, the precise rate of decay (with n) of the probability that a randomly chosen vertex of the unit cube $C = [-1, 1]^n$ lies in the halfspace determined by the hyperplane tangent to F^{β} at $\vec{x}$ which does not contain F^{β}; and (2) on a result of Montgomery-Smith [10], giving lower bounds for the tails of the distribution of Rademacher sums.

3 Comparing F^{α} to Q^{α}: the main lemma

Let $n \in \mathbb{N}$ and let $X_1, X_2, \ldots, X_n$ be independent and identically distributed ± 1 random variables, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with distribution $\mathbb{P}(X = 1) = \mathbb{P}(X = -1) = \frac{1}{2}$. For $t \in \mathbb{R}$, let

$$\varphi(t) := \mathbb{E}\left[e^{tX}\right] = \cosh(t) \tag{3.1}$$

be the common moment generating function of the X_i's, and set

$$\psi(t) := \log \varphi(t) = \log \cosh(t). \tag{3.2}$$

Finally, define $h : (-1, 1) \to \mathbb{R}$ by

$$h(x) := \tfrac{1}{2} \log \left(\frac{1+x}{1-x} \right). \tag{3.3}$$

It is easily checked that h is strictly convex and strictly increasing on $[0, 1)$. It is also easily checked that

$$f(x) = -\psi(h(x)) + x\, h(x) \quad \text{and} \quad f'(x) = h(x). \tag{3.4}$$

Given $x_1, \dots, x_n$ in $(-1, 1)$, set

$$t_i := h(x_i) \qquad (i \le n); \tag{3.5}$$

in the sequel, t_i and x_i will always be in this relationship. Observe that

$$x_i = \psi'(t_i) = \tanh(t_i) \qquad (i \le n). \tag{3.6}$$

Define a new probability measure $\mathbb{P}_{x_1,\dots,x_n}$ on $(\Omega, \mathcal{F})$, by

$$\mathbb{P}_{x_1,\dots,x_n}(A) := \mathbb{E}\left[\mathbf{1}_A \exp\left(\sum_{i=1}^n t_i X_i\right)\right] \prod_{i=1}^n [\varphi(t_i)]^{-1} \tag{3.7}$$

for $A \in \mathcal{F}$. The next lemma is verified by direct computation.

Lemma 3.1 *Under $\mathbb{P}_{x_1,\dots,x_n}$, the random variables $t_1X_1, \dots, t_nX_n$ are independent, with mean, variance and absolute central third moment given by*

$$\begin{aligned}
\mathbb{E}_{x_1,\dots,x_n}[t_iX_i] &= t_ix_i,\\
\mathbb{E}_{x_1,\dots,x_n}\left[t_i^2(X_i - x_i)^2\right] &= \frac{t_i^2}{\cosh^2(t_i)},\\
\mathbb{E}_{x_1,\dots,x_n}\left[|t_i(X_i - x_i)|^3\right] &= |t_i|^3 \frac{\cosh(2t_i)}{\cosh^4(t_i)},
\end{aligned}$$

respectively. □

Set

$$\sigma_n^2 := \sum_{i=1}^n \mathbb{E}_{x_1,\dots,x_n}\left[t_i^2(X_i - x_i)^2\right] = \sum_{i=1}^n \frac{t_i^2}{\cosh^2(t_i)} \tag{3.8}$$

and

$$S_n := \frac{1}{\sigma_n} \sum_{i=1}^n t_i(X_i - x_i), \tag{3.9}$$

and let $F_n : \mathbb{R} \to \mathbb{R}$ be the cumulative distribution function of the random variable S_n under the probability law $\mathbb{P}_{x_1,\dots,x_n}$:

$$F_n(x) := \mathbb{P}_{x_1,\dots,x_n}(S_n \le x) \qquad (x \in \mathbb{R}). \tag{3.10}$$

Write also μ_n for the probability measure on $\mathbb{R}$ defined by

$$\mu_n(-\infty, x] := F_n(x) \qquad (x \in \mathbb{R}). \tag{3.11}$$

Finally, set

$$\rho_n^{(3)} := \sum_{i=1}^n \mathbb{E}_{x_1,\dots,x_n}\left[|t_i(X_i - x_i)|^3\right] = \sum_{i=1}^n |t_i|^3 \frac{\cosh(2t_i)}{\cosh^4(t_i)}. \tag{3.12}$$

Since $\cosh(2y) \le 2\cosh^2(y)$, we have that

$$\frac{\rho_n^{(3)}}{\sigma_n^2} \le 2 \max_{1\le i\le n} |t_i|. \tag{3.13}$$

Notice also that

$$\mathbb{E}_{x_1,\ldots,x_n}[S_n] = 0 \quad \text{and} \quad \mathrm{Var}_{x_1,\ldots,x_n}[S_n] = 1.$$

By (3.7), we have that

$$\mathbb{P}\left(\sum_{i=1}^n t_i(X_i - x_i) \ge 0\right) = \mathbb{E}_{x_1,\ldots,x_n}\left[\mathbf{1}_{[0,\infty)}\left(\sum_{i=1}^n t_i(X_i - x_i)\right) \exp\left(-\sum_{i=1}^n t_i X_i\right)\right] \prod_{i=1}^n \varphi(t_i);$$

hence, using (3.4), (3.8), (3.9) and (3.11), we see that

$$\mathbb{P}\left(\sum_{i=1}^n t_i(X_i - x_i) \ge 0\right) = \int_{[0,\infty)} e^{-\sigma_n u}\, d\mu_n(u) \exp\left(\sum_{i=1}^n [\psi(t_i) - t_i x_i)]\right),$$

which then, upon using (3.4) and (3.5), yields that,

$$\mathbb{P}\left(\sum_{i=1}^n t_i(X_i - x_i) \ge 0\right) = \int_{[0,\infty)} e^{-\sigma_n u}\, d\mu_n(u) \exp\left(-\sum_{i=1}^n f(x_i)\right). \tag{3.14}$$

Next, write

$$\phi(x) := \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad (x \in \mathbb{R}) \tag{3.15}$$

for the standard gaussian density,

$$\Phi(x) := \int_{-\infty}^x \phi(y)\, dy \qquad (x \in \mathbb{R}) \tag{3.16}$$

for the standard gaussian c.d.f., and μ for the standard gaussian probability measure: $\mu(-\infty, x] := \Phi(x)$, $x \in \mathbb{R}$. By the Berry–Esseen theorem [5, Theorem XVI.5.2] and Lemma 3.1, one has

$$|F_n(x) - \Phi(x)| \le 6\, \frac{\rho_n^{(3)}}{\sigma_n^3} \tag{3.17}$$

for all x, whence by (3.13),

$$|F_n(x) - \Phi(x)| \le \frac{12}{\sigma_n} \max_{1\le i\le n} |t_i| \tag{3.18}$$

for all x.

Lemma 3.2 *The following inequality holds:*

$$\left| \int_{(0,\infty)} e^{-\sigma_n u} \, d\mu_n(u) - \int_{(0,\infty)} e^{-\sigma_n u} \, d\mu(u) \right| \le \frac{24}{\sigma_n} \max_{1 \le i \le n} |t_i|.$$

Proof. Since

$$\begin{aligned} \int_{(0,\infty)} e^{-\sigma_n u} \, d\mu_n(u) &= \int_0^\infty \mu_n \left(\{ u \colon e^{-\sigma_n u} \mathbf{1}_{(0,\infty)}(u) \ge r \} \right) \, dr \\ &= \int_0^1 \mu_n \left(0, -\sigma_n^{-1} \log r \right] \, dr \\ &= \int_0^1 \left[F_n \left(-\sigma_n^{-1} \log r \right) - F_n(0) \right] \, dr, \end{aligned}$$

and similarly

$$\int_0^\infty e^{-\sigma_n u} \, d\mu(u) = \int_0^1 \left[\Phi \left(-\sigma_n^{-1} \log r \right) - \Phi(0) \right] \, dr,$$

the result follows from (3.18). □

We will use the estimates

$$\frac{1}{x} m_1(x) \, e^{-x^2/2} \le \int_x^\infty \phi(u) \, du \le \frac{1}{x} m_2(x) \, e^{-x^2/2} \qquad (x > 0),$$

where

$$m_1(x) = \frac{1}{\sqrt{2\pi}} \frac{2x}{x + \sqrt{x^2+4}} \qquad \text{and} \qquad m_2(x) = \frac{1}{\sqrt{2\pi}} \frac{4x}{3x + \sqrt{x^2+8}} \tag{3.19}$$

(see [8, p. 17] and, for the upper estimate, [12]). Since

$$\int_0^\infty e^{-\sigma_n u} \, d\mu(u) = \frac{e^{\sigma_n^2/2}}{\sqrt{2\pi}} \int_0^\infty e^{-(\sigma_n + u)^2/2} \, du = e^{\sigma_n^2/2} \int_{\sigma_n}^\infty \phi(u) \, du,$$

it follows that

$$\frac{m_1(\sigma_n)}{\sigma_n} \le \int_0^\infty e^{-\sigma_n u} \, d\mu(u) \le \frac{m_2(\sigma_n)}{\sigma_n}. \tag{3.20}$$

Combining (3.20) with Lemma 3.2, we obtain the estimates

$$\frac{m_1(\sigma_n)}{\sigma_n} - \frac{24}{\sigma_n} \max_{1 \le i \le n} |t_i| \le \int_{(0,\infty)} e^{-\sigma_n u} \, d\mu_n(u) \le \frac{m_2(\sigma_n)}{\sigma_n} + \frac{24}{\sigma_n} \max_{1 \le i \le n} |t_i|. \tag{3.21}$$

Equation (3.14) then yields the following estimates:

Theorem 3.3 *Let $x_1, \dots, x_n \in (-1,1)$ and $t_i = h(x_i)$, $i = 1, \dots, n$. Then,*

$$\mathbb{P}\left(\sum_{i=1}^{n} t_i(X_i - x_i) \geq 0\right) \leq \frac{1}{\sigma_n} e^{-nF(\vec{x})} \left(m_2(\sigma_n) + 48 \max_{1\leq i\leq n} |t_i|\right) \tag{3.22}$$

and

$$\mathbb{P}\left(\sum_{i=1}^{n} t_i(X_i - x_i) \geq 0\right) \geq \frac{1}{\sigma_n} e^{-nF(\vec{x})} \left(m_1(\sigma_n) - 24 \max_{1\leq i\leq n} |t_i|\right). \tag{3.23}$$

Proof. Observe that the first factor on the right hand-side of (3.14) is

$$\int_{[0,\infty)} e^{-\sigma_n u}\, d\mu_n(u) \geq \int_{(0,\infty)} e^{-\sigma_n u}\, d\mu_n(u).$$

Hence, we obtain the second inequality (3.23) by combining (3.14) with (3.21). For the first inequality, first observe that

$$\int_{[0,\infty)} e^{-\sigma_n u}\, d\mu_n(u) = \int_{(0,\infty)} e^{-\sigma_n u}\, d\mu_n(u) + \mathbb{P}_{x_1,\dots,x_n}(S_n = 0),$$

and then use (3.18) to obtain,

$$\mathbb{P}_{x_1,\dots,x_n}(S_n = 0) \leq F_n(\epsilon) - F_n(-\epsilon) \leq \Phi(\epsilon) - \Phi(-\epsilon) + \frac{24}{\sigma_n} \max_{1\leq i\leq n} |t_i|$$

for all $\epsilon > 0$. Thus

$$\mathbb{P}_{x_1,\dots,x_n}(S_n = 0) \leq \frac{24}{\sigma_n} \max_{1\leq i\leq n} |t_i|, \tag{3.24}$$

and the first inequality (3.23) follows now as well, using (3.14), (3.21) and (3.24) this time. □

Corollary 3.4 *Let $\delta \in (0,1)$. If $x_1, \dots, x_n \in (-\delta, \delta)$ and $t_i = h(x_i)$, $i = 1, \dots, n$, then*

$$\mathbb{P}\left(\sum_{i=1}^{n} t_i(X_i - x_i) \geq 0\right) \leq \frac{1}{\sqrt{nF(\vec{x})}} e^{-nF(\vec{x})} \times \frac{\cosh(h(\delta))}{\sqrt{2}} \left(m_2\Big(h(\delta)(f(\delta))^{-1/2}\sqrt{nF(\vec{x})}\Big) + 48\, h(\delta)\right).$$

and

$$\mathbb{P}\left(\sum_{i=1}^{n} t_i(X_i - x_i) \geq 0\right) \geq \frac{1}{\sqrt{nF(\vec{x})}} e^{-nF(\vec{x})} \times \frac{\sqrt{f(\delta)}}{h(\delta)} \left(m_1\Big((\cosh(h(\delta)))^{-1}\sqrt{2nF(\vec{x})}\Big) - 24\, h(\delta)\right).$$

Proof. Recall (3.8). The function

$$g(t) = \frac{f(\tanh(t))}{t^2} = -\frac{1}{t^2}\log\cosh(t) + \frac{\tanh(t)}{t} \tag{3.25}$$

is strictly decreasing on $[0,\infty)$ and $\lim_{t\to 0} g(t) = \frac{1}{2}$ (see [1, Lemma 6.1]). It follows that

$$\frac{f(\delta)}{h^2(\delta)} t_i^2 \leq f(x_i) \leq \frac{1}{2} t_i^2 \tag{3.26}$$

for all $i \leq n$, since also h is increasing on $[0,1)$. Since $1 \leq \cosh^2(t_i) \leq \cosh^2(h(\delta))$, we get that

$$\frac{f(\delta)}{h^2(\delta)}\,\sigma_n^2 \leq nF(\vec{x}) \leq \frac{\cosh^2(h(\delta))}{2}\,\sigma_n^2. \tag{3.27}$$

Finally, we also have that

$$\max_{1\leq i\leq n} |t_i| \leq h(\delta). \tag{3.28}$$

Inserting these estimates into the estimates of Theorem 3.3, and using the fact that m_1 and m_2 are increasing on $[0,\infty)$, concludes the proof. □

The upper bound (3.22), and its equivalent in Corollary 3.4, will not be used in the sequel and are only given for completeness. Notice, however, that these bounds subsume Lemma 2.1.

Corollary 3.5 *There exist $\gamma \in (0,1)$ and $k = k(\gamma) \in \mathbb{N}$ with the following property: For every $n \in \mathbb{N}$, if $x_1, \dots, x_n \in (-\gamma, \gamma)$ are such that $\sum_{i=1}^n f(x_i) \geq k(\gamma)$, and if $t_i = h(x_i)$, $i = 1, \dots, n$, then*

$$\mathbb{P}\left(\sum_{i=1}^n t_i(X_i - x_i) \geq 0\right) \geq \frac{\sqrt{f(\gamma)}}{10\,h(\gamma)} \times \frac{1}{\sqrt{nF(\vec{x})}}\, e^{-nF(\vec{x})}. \tag{3.29}$$

Proof. First choose $\gamma \in (0,1)$ so that $24h(\gamma) \leq (2\sqrt{2\pi})^{-1}$; this is possible because $\lim_{\delta\to 0} h(\delta) = 0$.

We know that m_1 increases to $(2\pi)^{-1/2}$ as $x \to \infty$; so, there exists $k = k(\gamma) \in \mathbb{N}$ such that

$$m_1\left(\frac{\sqrt{2k(\gamma)}}{\cosh\big(h(\gamma)\big)}\right) \geq \frac{5}{6\sqrt{2\pi}}.$$

From the second assertion of Corollary 3.4 we easily check (3.29) for all $x_i \in (-\gamma, \gamma)$ with $nF(\vec{x}) = \sum_{i=1}^n f(x_i) \geq k(\gamma)$. □

We shall also make essential use of the following result of Montgomery-Smith from [10]:

Lemma 3.6 *There exists a universal constant c, such that, for all $n \in \mathbb{N}$ and any $s_1, \ldots, s_n \in \mathbb{R}$, the inequality*

$$\mathbb{P}\left(\sum_{i=1}^{n} s_i X_i \geq c^{-1} a \|\vec{s}\|_2\right) \geq c^{-1} e^{-ca^2}$$

holds for all $a > 0$ with $a \leq \|\vec{s}\|_2 / \|\vec{s}\|_\infty$, where $\vec{s} = (s_1, \ldots, s_n)$. □

Definition 3.7 In the sequel we fix a constant $\gamma \in (0,1)$ which satisfies Corollary 3.5 and also $\gamma \leq \tanh(c^{-1})$, where c is the universal constant of Lemma 3.6. For example, since one can have $c = 4 \log 12$ in Lemma 3.6, we may take

$$\gamma = \min\left\{\tanh\left(\frac{1}{48\sqrt{2\pi}}\right), \tanh\left(\frac{1}{4\log 12}\right)\right\} = \tanh\left(\frac{1}{48\sqrt{2\pi}}\right). \tag{3.30}$$

We shall also use the following lemma (see [1, Lemma 8.2]).

Lemma 3.8 *Let $\gamma \in (0,1)$ and $s_i > 0$, $i = 1, \ldots, m$. Then, for $m > 2/(1-\gamma)$,*

$$\begin{aligned}\mathbb{P}\left(\sum_{i=1}^{m} s_i (X_i - \gamma) \geq 0\right) \quad \geq \quad & \sqrt{\frac{2}{\pi(1-\gamma^2)}\,\frac{1-\gamma-2m^{-1}}{1+\gamma+2m^{-1}}} \\ & \times \exp\left(\frac{1}{12m+1} - \frac{1}{12m}\,\frac{4}{1-(\gamma+2m^{-1})^2}\right) \\ & \times \frac{1}{m^{3/2}} \exp(-mf(\gamma)).\end{aligned}$$

In particular, for $\gamma \leq \tanh\left((48\sqrt{2\pi})^{-1}\right)$ and $m > 2$,

$$\mathbb{P}\left(\sum_{i=1}^{m} s_i (X_i - \gamma) \geq 0\right) \geq c(\gamma)\, m^{-3/2}\, e^{-mf(\gamma)},$$

where $c(\gamma) > 0$ is given by

$$c(\gamma) = \sqrt{\frac{2}{\pi(1-\gamma^2)}\,\frac{1-3\gamma}{5+3\gamma}}\ \exp(-[9-(3\gamma+2)^2]^{-1}).$$

Proof. This is proved as Lemma 8.2 of [1]. The argument there gives

$$\mathbb{P}\left(\sum_{i=1}^{m} s_i (X_i - \gamma) \geq 0\right) \geq \frac{1}{m}\,\frac{1}{2^m} \sum_{\frac{\gamma+1}{2} \leq \frac{k}{m} \leq 1} \binom{m}{k} \geq \frac{1}{m}\,\frac{1}{2^m} \binom{m}{k_\gamma},$$

where $k_\gamma := \lceil m(\gamma+1)/2 \rceil$ is the least integer $\geq m(\gamma+1)/2$; in particular,

$$\frac{k_\gamma}{m} < \frac{1+\gamma+2m^{-1}}{2}.$$

Now using H. E. Robbins' fine form of the Stirling approximation given in [4, II, (9.15)], and the fact that

$$f(\gamma + 2m^{-1}) \leq f(\gamma) + 2m^{-1}h(\gamma + 2m^{-1})$$

(by the mean value theorem and the monotonicity of $f' = h$), yields the result. □

Theorem 3.9 *There exists $n_0 = n_0(\gamma) \in \mathbb{N}$ with the following property: If $n \geq n_0$ and $4\log n/n \leq \alpha \leq \log 2$, then*

$$F^{\alpha-\varepsilon_2} \cap \gamma C \subseteq Q^\alpha$$

for some $\varepsilon_2 \leq 3\log n/n$.

Proof. Fix $\varepsilon_2 = 3\log n/n$. We need to check that $q(\vec{x}) \geq \exp(-\alpha n)$ for every $\vec{x}$ in $F^{\alpha-\varepsilon_2} \cap \gamma C$. It suffices to prove that

$$\mathbb{P}\big(\vec{X} \in H\big) \geq \exp(-\alpha n) \tag{3.31}$$

for every halfspace H touching $F^{\alpha-\varepsilon_2} \cap \gamma C$. In the proof of [1, Lemma 4.3] it is explained that, for any such halfspace H, there exists $\vec{x}$ on the bounding hyperplane of H such that $F(\vec{x}) = \alpha - \varepsilon_2$. By symmetry we may assume that $\vec{x} = (x_1, \ldots, x_n)$ with $0 < x_1 \leq x_2 \leq \cdots \leq x_n$.

There exists $n_1 \in \{1, \ldots, n\}$ such that $x_{n_1} < \gamma$ and $x_{n_1+1} = \gamma$. As in [1, Lemma 4.3] we set $\vec{t} = (t_1, \ldots, t_n) = n\,\nabla F(\vec{x})$ and write $\vec{t_*} = (t_1^*, \ldots, t_n^*)$ for the normal to the bounding hyperplane of H. We may assume that $\vec{t_*}$ is in the relative interior of the normal cone to $F^{\alpha-\varepsilon_2} \cap \gamma C$ at $\vec{x}$, whence

$$t_i^* = t_i = f'(x_i) \quad \text{if} \quad i \leq n_1 \quad \text{and} \quad t_i^* > t_i = f'(\gamma) \quad \text{if} \quad i > n_1.$$

Write

$$\begin{aligned}
\mathbb{P}\left(\sum_{i=1}^{n} t_i^*(X_i - x_i) \geq 0\right) &= \mathbb{P}\left(\sum_{i=1}^{n_1} t_i(X_i - x_i) + \sum_{i=n_1+1}^{n} t_i^*(X_i - \gamma) \geq 0\right) \\
&\geq \mathbb{P}\left(\sum_{i=1}^{n_1} t_i(X_i - x_i) \geq 0\right)\ \mathbb{P}\left(\sum_{i=n_1+1}^{n} t_i^*(X_i - \gamma) \geq 0\right).
\end{aligned}$$

We estimate the second probability in the last product using Lemma 3.8:

$$\mathbb{P}\left(\sum_{i=n_1+1}^{n} t_i^*(X_i - \gamma) \geq 0\right) \geq \exp\left(-(n-n_1)f(\gamma) - \tfrac{3}{2}\log(n-n_1) - c_1(\gamma)\right). \tag{3.32}$$

To estimate the first probability we distinguish two cases:

Case 1: $\sum_{i=1}^{n_1} f(x_i) \geq k(\gamma)$. We may then use Corollary 3.5 to estimate the first probability:

$$\mathbb{P}\left(\sum_{i=1}^{n_1} t_i(X_i - x_i) \geq 0\right) \geq \exp\left(-\sum_{i=1}^{n_1} f(x_i) - \tfrac{1}{2}\log\sum_{i=1}^{n_1} f(x_i) - c_2(\gamma)\right). \tag{3.33}$$

Combining (3.32) and (3.33) we obtain

$$\begin{aligned}
\mathbb{P}\left(\sum_{i=1}^{n} t_i^*(X_i - x_i) \geq 0\right) &\geq \exp\left(-\sum_{i=1}^{n_1} f(x_i) - \tfrac{1}{2}\log\sum_{i=1}^{n_1} f(x_i)\right) \\
&\quad \times \exp\left(-(n-n_1)f(\gamma) - \tfrac{3}{2}\log(n-n_1) - c(\gamma)\right) \\
&\geq \exp\left(-\sum_{i=1}^{n} f(x_i) - 2\log n - c(\gamma)\right) \\
&= \exp\left(-(\alpha - \varepsilon_2)n - 2\log n - c(\gamma)\right) \\
&\geq \exp(\alpha n),
\end{aligned}$$

provided n is large enough to have that $\log n \geq c(\gamma)$.

Case 2: $\sum_{i=1}^{n_1} f(x_i) < k(\gamma)$. In this case we use Lemma 3.6 to estimate the first probability. We have that

$$\mathbb{P}\left(\sum_{i=1}^{n_1} t_i(X_i - x_i) \geq 0\right) = \mathbb{P}\left(\sum_{i=1}^{n_1} t_i X_i \geq c^{-1} a \sqrt{\sum_{i=1}^{n_1} t_i^2}\right), \tag{3.34}$$

with

$$a = c\,\frac{\sum_{i=1}^{n_1} t_i x_i}{\sqrt{\sum_{i=1}^{n_1} t_i^2}}.$$

To use Lemma 3.6 we need to check that

$$a \leq \frac{\sqrt{\sum_{i=1}^{n_1} t_i^2}}{\max_{1\leq i\leq n_1} t_i}.$$

The function h is convex on $[0,1)$ and its derivative at $x = 0$ is equal to 1; hence $x \leq h(x)$ for all $x \in [0,1)$. It follows that $\sum_{i=1}^{n_1} t_i x_i \leq \sum_{i=1}^{n_1} t_i^2$, by (3.5). Therefore,

$$a \leq c\sqrt{\sum_{i=1}^{n_1} t_i^2} \leq c\,h(\gamma)\,\frac{\sqrt{\sum_{i=1}^{n_1} t_i^2}}{\max\limits_{1\leq i\leq n_1} t_i}, \tag{3.35}$$

since also $t_i = h(x_i) \leq h(\gamma)$ for all i, by the monotonicity of h. By (3.30) and the monotonicity of h, $c\,h(\gamma) \leq 1$ and a satisfies the condition of Lemma 3.6. Lemma 3.6 and (3.34) then yield the bound

$$\mathbb{P}\left(\sum_{i=1}^{n_1} t_i(X_i - x_i) \geq 0\right) \geq c^{-1}e^{-ca^2} \geq \frac{1}{c}\exp\left(-c^3\,\frac{h^2(\gamma)}{f(\gamma)}\,\sum_{i=1}^{n_1} f(x_i)\right), \tag{3.36}$$

the last inequality by the first inequality in (3.35) and (3.26). Then, (3.36) and

(3.32) yield the bound

$$
\begin{aligned}
&\mathbb{P}\left(\sum_{i=1}^{n} t_i(X_i - x_i) \geq 0\right) \\
&\quad \geq \exp\left(-(n-n_1)f(\gamma) - \tfrac{3}{2}\log(n-n_1) - c_1(\gamma) - c_3(\gamma)\sum_{i=1}^{n_1} f(x_i) - c_4(\gamma)\right) \\
&\quad \geq \exp\left(-\sum_{i=1}^{n} f(x_i) - \tfrac{3}{2}\log(n-n_1) - c_1(\gamma) - |1-c_3(\gamma)|k(\gamma) - c_4(\gamma)\right) \\
&\quad = \exp\left(-(\alpha-\varepsilon_2)n - \tfrac{3}{2}\log(n-n_1) - C(\gamma)\right) \\
&\quad \geq \exp\left(-\alpha n\right),
\end{aligned}
$$

provided again that n is large enough to have $\log n \geq C(\gamma)$.

Therefore, in both Cases we have that

$$\mathbb{P}\left(\sum_{i=1}^{n} t_i^*(X_i - x_i) \geq 0\right) \geq \exp\big(-\alpha n\big)$$

for $n \geq n_0(\gamma)$, and this completes the proof. □

4 Weakly sandwiching K_N

The families $\{Q^\beta\}$ and $\{F^\beta\}$ are related to the behaviour of the random polytope K_N. Fix N with $n < N \leq 2^n$ and define α by the equation $N = e^{\alpha n}$. In other words,

$$\alpha = \frac{\log N}{n}. \tag{4.1}$$

In [1, Lemma 4.2] (see also [3, Lemma 2.1(b)]), it is proved that, for some small $\varepsilon_1(N,n) \in (0,1)$, the probability $\mathrm{Prob}\big(Q^{\alpha-\varepsilon_1} \not\subseteq K_N\big)$ is very small if n is large enough. On the other hand, in [1, Lemma 4.4] it is proved that, for some small $\varepsilon_3(N,n) \in (0,1)$, at least half of the surface area of $F^{\alpha+\varepsilon_3}$ lying in $\frac{1}{10}C$ is missed by the typical K_N. In view of Theorem 3.9 (or [1, Lemma 4.3]), this means that $K_N \cap \frac{1}{10}C$ is "weakly sandwiched" between $F^{\alpha-(\varepsilon_1+\varepsilon_2)} \cap \frac{1}{10}C$ and $F^{\alpha+\varepsilon_3}$. In this Section we provide new estimates for the parameters ε_1 and ε_3 in the two statements above.

Lemma 4.1 *Assume that $e^2 n \leq N \leq 2^n$. If n is sufficiently large, there exists $\varepsilon_1 \leq 3\log n/n$ such that*

$$\mathrm{Prob}\big(K_N \supseteq Q^{\alpha-\varepsilon_1}\big) > 1 - 2^{-(n-1)}. \tag{4.2}$$

Proof. Let $0 < \beta < \alpha$. The argument in [3, Lemma 2.1(b)] gives

$$1 - \mathrm{Prob}\big(K_N \supseteq Q^\beta\big) \le \binom{N}{n} 2^{-(N-n)} + 2\binom{N}{n}\big(1 - e^{-\beta n}\big)^{N-n}. \tag{4.3}$$

We will bound both terms on the right handside by 2^{-n}. Since $\binom{N}{n} \le (eN/n)^n$, we only need to check that

$$\left(\frac{eN}{n}\right)^n 2^{-(N-n)} < 2^{-n} \tag{4.4}$$

in order to bound the first term. This is equivalent to

$$(N - 2n)\log 2 > n \log\left(\frac{eN}{n}\right). \tag{4.5}$$

Since $eN/n \ge e^3$ and the function $g(x) = \log x/x$ is decreasing for $x \ge e$, we have that

$$\log\left(\frac{eN}{n}\right) \le \frac{3eN}{e^3 n}. \tag{4.6}$$

So, it suffices to check that

$$3e^{-2}N < (\log 2)N - 2(\log 2)n. \tag{4.7}$$

But, $e^2 \log 2 > 2\log 2 + 3$ and $N \ge e^2 n$; therefore

$$(\log 2 - 3e^{-2})N \ge (e^2 \log 2 - 3)n > 2(\log 2)n, \tag{4.8}$$

which proves (4.4).

Next, we will show that

$$2\binom{N}{n}\big(1 - e^{-\beta n}\big)^{N-n} < 2^{-n}, \tag{4.9}$$

provided that n is large enough and $\alpha - \beta \ge 3\log n/n$. Since $1 - x \le e^{-x}$, it suffices to check that

$$\left(\frac{4eN}{n}\right)^n \exp\big(-e^{-\beta n}(N-n)\big) < 1. \tag{4.10}$$

Write $\beta = \alpha - \varepsilon$, where $\varepsilon > 0$. Then, $e^{-\beta n} = e^{\varepsilon n}/N$. Since $n\log(4eN/n) \le n^2$ (assume that $n \ge 12$) and $(N-n)/N \ge \frac{1}{2}$, we want

$$2n^2 \le e^{\varepsilon n}. \tag{4.11}$$

This is satisfied if $\varepsilon \ge 3\log n/n$.

From (4.4) and (4.9) we have the lemma. $\square$

Lemma 4.2 *Let n be large enough and assume that $\alpha < \log 2 - 12n^{-1}$. There exists $\varepsilon_3 \leq 6/n$ such that*

$$\text{(4.12)} \qquad \operatorname{Prob}\big(\, |\partial(F^{\alpha+\varepsilon_3}) \cap \gamma C \cap K_N| \geq \tfrac{1}{2}|\partial(F^{\alpha+\varepsilon_3}) \cap \gamma C| \, \big) \leq \tfrac{1}{100}.$$

Proof. Let $\beta > \alpha$ and write $\beta = \alpha + \varepsilon_3$ for some $\varepsilon_3 > 0$. Let $\vec{x}$ be on the boundary of F^β. If H is a halfspace containing $\vec{x}$, and if $\vec{x} \in K_N = \operatorname{conv}\{\vec{X}_1, \dots, \vec{X}_N\}$, then there exists $i \leq N$ such that $\vec{X}_i \in H$ (otherwise we would have $\vec{x} \in K_N \subseteq F$, where F is the complementary halfspace). We write

$$\begin{aligned}
\operatorname{Prob}\big(\vec{x} \in K_N\big) &= \operatorname{Prob}\big(\, \vec{x} \in \operatorname{conv}\{\vec{X}_1, \dots, \vec{X}_N\} \, \big) \\
&\leq \operatorname{Prob}\big(\, \vec{X}_i \in H \text{ for some } 1 \leq i \leq N \, \big) \\
&\leq \sum_{i=1}^{N} \operatorname{Prob}\big(\, \vec{X}_i \in H \, \big) \\
&= N \operatorname{Prob}\big(\vec{X} \in H\big).
\end{aligned}$$

Since $H \ni \vec{x}$ was arbitrary,

$$\text{(4.13)} \qquad \operatorname{Prob}\big(\vec{x} \in K_N\big) \leq N \inf\big\{\operatorname{Prob}\big(\vec{X} \in H\big) \colon H \ni \vec{x}\big\} = N q(\vec{x}).$$

From Lemma 2.1 we have

$$\text{(4.14)} \qquad N q(\vec{x}) \leq N \exp(-nF(\vec{x})) = N \exp(-\alpha n - \varepsilon_3 n) = \exp(-\varepsilon_3 n) < \tfrac{1}{200}$$

if $\varepsilon_3 \geq 6/n$. Now

$$\text{(4.15)} \quad \mathbb{E}\,|(\partial(F^\beta) \cap \gamma C \cap K_N| \leq \int_{\partial(F^\beta) \cap \gamma C} \operatorname{Prob}\big(\vec{x} \in K_N\big)\, d\vec{x} \leq \tfrac{1}{200}|\partial(F^\beta) \cap \gamma C|.$$

Therefore

$$\text{(4.16)} \qquad \operatorname{Prob}\big(\, |(\partial(F^\beta) \cap \gamma C \cap K_N| \geq \tfrac{1}{2}|\partial(F^\beta) \cap \gamma C| \, \big) \leq 10^{-2},$$

by Markov's inequality. □

5 Proof of the Theorem

We follow the idea of Bárány and Pór. We need the following two facts from [1]:

Lemma 5.1 *For every $\gamma \in (0,1)$, there exists a constant $c(\gamma) > 0$, such that if n is large enough and $\beta \leq c(\gamma)/\log n$, then*

$$\text{(5.1)} \qquad |\partial(F^\beta) \cap \gamma C| \geq \tfrac{1}{2}\big(1 - \gamma^2\big)^{n-1} (2\beta n)^{(n-1)/2} |S^{n-1}|.$$

Proof. We sketch the argument from [1, Lemma 5.1] in order to make the necessary adjustments. We first estimate the product curvature $\kappa(\vec{x})$ of the surface $F(\vec{x})=\beta$ (the formula which appears in [1] is not exact; we would like to express our gratitude to I. Bárány, V. Kaibel and R. Mechtel for kindly pointing out this point). Let $\nu(\vec{x})=\nabla F(\vec{x})/\|\nabla F(\vec{x})\|_2$ be the outward unit normal vector of F^β at $\vec{x}$. Following [11, Section 2.5], we write $T_{\vec{x}}F^\beta$ for the tangent space of F^β at $\vec{x}$, and consider the Weingarten map $W_{\vec{x}}:T_{\vec{x}}F^\beta\to T_{\vec{x}}F^\beta$. This is the restriction to $T_{\vec{x}}F^\beta$ of the differential $D_{\vec{x}}$ of the map $\vec{x}\mapsto\nu(\vec{x})$. Then $W_{\vec{x}}$ is symmetric and positive definite, therefore

$$\kappa(\vec{x})=\det W_{\vec{x}}\leq\left(\frac{\mathrm{trace}(W_{\vec{x}})}{n-1}\right)^{n-1} \tag{5.2}$$

by the arithmetic-geometric means inequality. Let $(a_{ij})_{i,j=1}^n$ denote the matrix of $D_{\vec{x}}$ with respect to the standard basis of $\mathbb{R}^n$. As is well known, and readily verified by direct computation, $\nu(\vec{x})$ is an eigenvector of the adjoint of $D_{\vec{x}}$, with corresponding eigenvalue 0; it follows from this and the fact that the eigenvalues of $W_{\vec{x}}$ are also eigenvalues of $D_{\vec{x}}$ and none of them is zero, that $\mathrm{trace}(W_{\vec{x}})=\mathrm{trace}(D_{\vec{x}})$. A simple calculation also shows that

$$a_{ii}=\frac{f''(x_i)\big(\|n\nabla F(\vec{x})\|_2^2-(f'(x_i))^2\big)}{\|n\nabla F(\vec{x})\|_2^3}=\frac{h'(x_i)\big(\|\,\vec{t}\,\|_2^2-(h(x_i))^2\big)}{\|\,\vec{t}\,\|_2^3}. \tag{5.3}$$

It follows that, if $\vec{x}\in\partial(F^\beta)\cap\gamma C$, then

$$\begin{aligned}\frac{\mathrm{trace}(W_{\vec{x}})}{n-1}=\frac{\mathrm{trace}(D_{\vec{x}})}{n-1} &= \sum_{i=1}^n\frac{h'(x_i)\big(\|\,\vec{t}\,\|_2^2-(h(x_i))^2\big)}{(n-1)\|\,\vec{t}\,\|_2^3}\\ &\leq h'(\gamma)\frac{n\|\,\vec{t}\,\|_2^2-\sum_{i=1}^n t_i^2}{(n-1)\|\,\vec{t}\,\|_2^3}\\ &= \frac{h'(\gamma)}{\|\,\vec{t}\,\|_2},\end{aligned}$$

and (5.2) shows that

$$\frac{1}{\kappa(\vec{x})}\geq\|\,\vec{t}\,\|_2^{n-1}(1-\gamma^2)^{n-1}. \tag{5.4}$$

Since also $2f(x_i)\leq t_i^2$, by (3.26), we finally have that

$$\frac{1}{\kappa(\vec{x})}\geq\big(1-\gamma^2\big)^{n-1}(2n\beta)^{(n-1)/2}. \tag{5.5}$$

For every $\vec{\theta}\in S^{n-1}$ we write $\vec{x}(\vec{\theta},\beta)$ for the point on the boundary of F^β for which $\vec{t}(\vec{\theta},\beta)=n\,\nabla F(\vec{x}(\vec{\theta},\beta))$ is a positive multiple of $\vec{\theta}$. This point is well-defined

and unique if $0 < \beta < |\mathrm{supp}(\vec{\theta})|\,(\log 2)/n$ (see [1, Lemma 6.2]). The argument given in [1, Lemma 6.3] shows that if

$$M = \left\{ \vec{\theta} \in S^{n-1} : \sqrt{\frac{n}{3\log n}}\ \vec{\theta} \in C \right\} \tag{5.6}$$

and if $\beta < c\gamma^2/\log n$, then for every $\vec{\theta} \in M$ we have $\vec{x}(\vec{\theta}, \beta) \in \gamma C$ (observe that if $\vec{\theta} \in M$, we also have $|\mathrm{supp}(\vec{\theta})| \geq n/(3\log n)$, and therefore $\beta < |\mathrm{supp}(\vec{\theta})|\,(\log 2)/n$ if $\beta < c\gamma^2/\log n$). A standard computation on the area of spherical caps shows that

$$|M| \geq \tfrac{1}{2}|S^{n-1}|. \tag{5.7}$$

Then, we can write

$$|\partial(F^\beta) \cap \gamma C| = \int_{S^{n-1}} \frac{1}{\kappa(\vec{x})}\, d\vec{\theta} \geq \int_M \frac{1}{\kappa(\vec{x})}\, d\vec{\theta} \geq \tfrac{1}{2}\bigl(1-\gamma^2\bigr)^{n-1}(2n\beta)^{(n-1)/2}|S^{n-1}| \tag{5.8}$$

as claimed. □

Lemma 5.2 *Let $\gamma \in (0, 0.1)$ and assume that $\beta + \varepsilon < \log 2$. If H is a halfspace which is disjoint from $F^\beta \cap \gamma C$, then*

$$|\partial(F^{\beta+\varepsilon}) \cap \gamma C \cap H| \leq (3\varepsilon n)^{(n-1)/2}|S^{n-1}|. \tag{5.9}$$

Proof. Completely similar to the one in [1, Lemma 5.2]. □

Proof of Theorem 1.2: Assume that n is large enough and set $b = c(\gamma)$, where $c(\gamma) > 0$ is the constant in Lemma 5.1.

Given N with $n^8 \leq N \leq \exp\bigl(bn/\log n\bigr)$, let $\alpha = \log N/n$ and let $\gamma \in (0,1)$ be the constant in (3.30). From Lemma 4.1 there exists $\varepsilon_1 \leq 3\log n/n$ such that

$$K_N \supset Q^{\alpha-\varepsilon_1} \tag{5.10}$$

with probability greater than $1 - 2^{-n+1}$. Then, Theorem 3.9 shows that

$$K_N \supset F^{\alpha-\varepsilon_1-\varepsilon_2} \cap \gamma C \tag{5.11}$$

with probability greater than $1 - 2^{-n+1}$, where $\varepsilon_2 \leq 3\log n/n$. Finally, Lemma 4.2 shows that there exists $\varepsilon_3 \leq 6/n$ such that K_N satisfies

$$|\partial(F^{\alpha+\varepsilon_3}) \cap \gamma C \cap K_N| \leq \tfrac{1}{2}|\partial(F^{\alpha+\varepsilon_3}) \cap \gamma C| \tag{5.12}$$

with probability greater than $1 - 10^{-2}$. Thus, there exists a 0/1 polytope K_N which satisfies all the above.

Next, apply Lemma 5.2 with $\beta = \alpha - \varepsilon_1 - \varepsilon_2$ and $\varepsilon = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$: If A is a facet of K_N and H_A is the corresponding halfspace (which has interior disjoint from K_N), then

$$|\partial(F^{\alpha+\varepsilon_3}) \cap \gamma C \cap H_A| \leq \bigl(3(\varepsilon_1+\varepsilon_2+\varepsilon_3)n\bigr)^{(n-1)/2}\,|S^{n-1}|. \tag{5.13}$$

It follows that

$$\begin{aligned} f_{n-1}(K_N)\left(3(\varepsilon_1+\varepsilon_2+\varepsilon_3)n\right)^{(n-1)/2}|S^{n-1}| &\geq \sum_A |\partial(F^{\alpha+\varepsilon_3})\cap\gamma C\cap H_A| \\ &= \left|\left(\partial(F^{\alpha+\varepsilon_3})\cap\gamma C\right)\setminus K_N\right| \\ &\geq \tfrac{1}{2}|\partial(F^{\alpha+\varepsilon_3})\cap\gamma C|. \end{aligned}$$

Now apply Lemma 5.1 with $\beta=\alpha+\varepsilon_3$ to get

$$f_{n-1}(K_N)\left(3(\varepsilon_1+\varepsilon_2+\varepsilon_3)n\right)^{(n-1)/2} \geq \tfrac{1}{2}\left((1-\gamma^2)\sqrt{2\alpha n}\right)^{n-1}. \tag{5.14}$$

Since $\alpha n=\log N$ and $(\varepsilon_1+\varepsilon_2+\varepsilon_3)n\leq 12\log n$, this shows that

$$f_{n-1}(K_N)\geq\left(\frac{c_1(\gamma)\log N}{\log n}\right)^{n/2} \tag{5.15}$$

and the proof is complete. □

Proof of Theorem 1.1: We can apply Theorem 1.2 with $N\geq\exp\left(c_1 n/\log n\right)$ where $c_1>0$ is an absolute constant. This shows that there exists a 0/1 polytope P in $\mathbb{R}^n$ with

$$f_{n-1}(P)\geq\left(\frac{c_2 n}{\log^2 n}\right)^{n/2}, \tag{5.16}$$

which is our lower bound for $g(n)$. □

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